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Eg. $y = f(x) = x^3$ $\quad f: (0,\infty) \to (0,\infty).$

f is 1-1 $\quad x = f^{-1}(y) = \sqrt[3]{y}.$

$$\frac{dy}{dx} = 3x^2. \qquad \frac{df^{-1}}{dy} = \frac{d}{dy} y^{\frac{1}{3}} = \frac{1}{3} y^{-\frac{2}{3}}$$

alternatively

$$\frac{df^{-1}}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3x^2}.$$

Now differentiate both sides of this equation term by term with respect to x:

$$\begin{aligned} \frac{d}{dx}[x^2 f(x) + (f(x))^2] &= \frac{d}{dx}[x^3] \\ \rightsquigarrow \left[x^2 \frac{df}{dx} + f(x)\frac{d}{dx}(x^2)\right] + 2f(x)\frac{df}{dx} &= 3x^2. \end{aligned} \tag{5.1}$$

Thus, we have

$$\begin{aligned} x^2\frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} &= 3x^2 \\ \rightsquigarrow [x^2 + 2f(x)]\frac{df}{dx} &= 3x^2 - 2xf(x) \\ \rightsquigarrow \frac{dy}{dx} &= \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)}. \end{aligned} \tag{5.2}$$

Finally, replace $f(x)$ by y to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}.$$

■

Remark. By default, $\frac{dy}{dx}$ is regarded as a function of x, and we want an expression for $\frac{dy}{dx}$ in terms of x only. However, sometimes it is difficult to express y in terms of x explicitly. In this case it'll be specified in the test or homework question that it is ok to leave the answer for y' as a function of both x and y. Or, sometimes finding the value for y' is only an intermediate step in solving the problem. If the values of x and y are known, one may directly plug in these values to the expression of y' in x and y, without going through an explicit formula for y' in x.

Summary: Carrying out Implicit Differentiation

Suppose an equation defines y implicitly as a differentiable function of x. To find $\dfrac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x, and use the chain rule when differentiating terms containing y.
2. Solve the differentiated equation algebraically for $\dfrac{dy}{dx}$ in terms of x and y.

Example 5.1.3. Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

1. Compute $\dfrac{dy}{dx}$. (It is ok to leave the answer as a function of both x and y.)
2. Find the slope of the tangent line to the curve at $(4, 2)$.

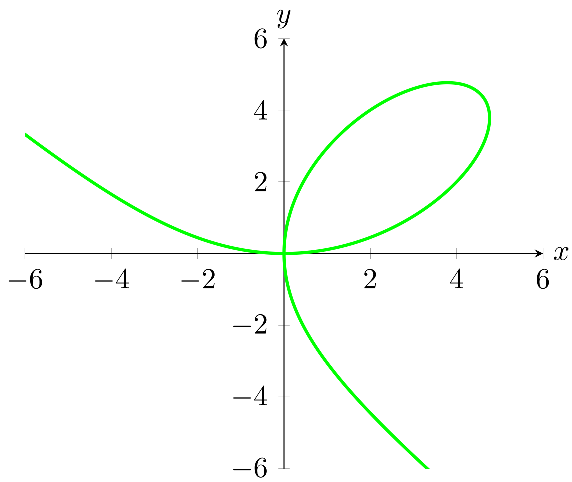


Figure 5.1: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Solution. Starting with

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

Applying the sum rule, we see that

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of the terms above in turn. To begin,

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand, $\frac{d}{dx}y^3$ is treated somewhat differently. Here, viewing $y = y(x)$ as an implicit function of x, we have by the chain rule that

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(y(x))^2 \cdot y'(x) \\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Consider the final term $\frac{d}{dx}(9xy)$. Regarding $y = y(x)$ again as an implicit function, we have:

$$\begin{aligned}\frac{d}{dx}(9xy) &= 9\frac{d}{dx}\big(x \cdot y(x)\big)\\ &= 9\left(x \cdot y'(x) + y(x)\right)\\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting all the above together, we get:

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

Now we solve the preceding equation for $\frac{dy}{dx}$. Write

$$\begin{aligned} & 3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y\\ \iff\quad & 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2\\ \iff\quad & \frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2\\ \iff\quad & \frac{dy}{dx} = \frac{9y-3x^2}{3y^2-9x} = \frac{3y-x^2}{y^2-3x}.\end{aligned}$$

For the second part of the problem, we simply plug in $x = 4$ and $y = 2$ to the last formula above to conclude that the slope of the tangent line to the curve at $(4, 2)$ is $\frac{5}{4}$. See Figure 5.2. ■

Example 5.1.4. Let L be the curve in the $x - y$ plane defined by $x^2 + y^2 + e^{xy} = 2$. Use L to implicitly define a function $y = y(x)$. Find $y'(x)$ at $x = 1$ and the tangent line to the curve L at $(1, 0)$.

Solution. (Note: In this case, there is no good explicit formula for the function $y(x)$.) Differentiate the equation $x^2 + y^2 + e^{xy} = 2$ on both sides with respect to x. We get:

$$\begin{aligned}2x + 2yy' + e^{xy}(y + xy') = 0,\\ \rightsquigarrow y' = -\frac{2x + e^{xy}y}{2y + e^{xy}x}.\end{aligned}$$

So, $y(1) = 0$ and $y'|_{x=1)} = -2$.

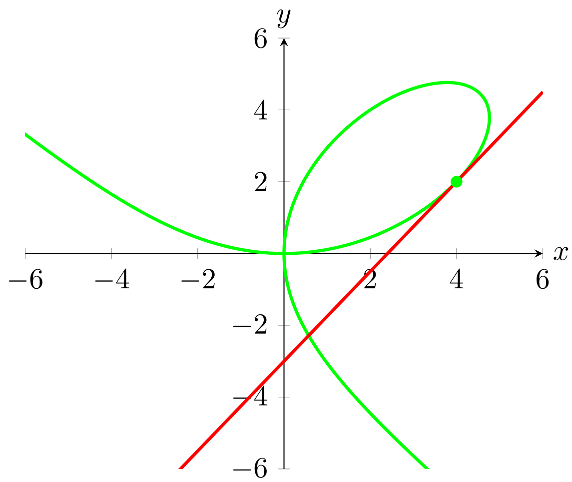


Figure 5.2: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.

Thus, the equation of the tangent line to L at $(x, y) = (1, 0)$ is:

$$\begin{aligned} y - 0 &= -2(x-1), \quad \text{or} \\ y &= -2x + 2. \end{aligned}$$

■

5.1.2 Differentiating Inverse Functions

Definition 5.1.1. Consider a function $f : A \to B$, where A is the domain, and B is the ~~range~~. co domain.

The function f is said to be *injective* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for any $x_1, x_2 \in A$. The function f is said to be *surjective* or *onto* if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$. The function f is said to be *bijective* or *one to one* if it is both injective and surjective.

If f is one-to-one, then the inverse function, denoted $f^{-1} : B \to A$, is defined by

$\forall y \in B,$

$$x = f^{-1}(y) \quad \text{if } y = f(x).$$

E.g. $f(x) = x$. domain $= \mathbb{R} =$ codomain
is one to one.
$f^{-1}(y) = y$

Remark.

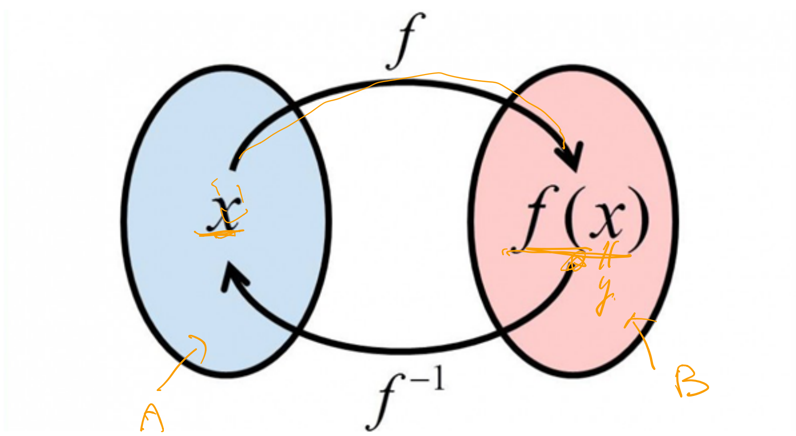
1. $f^{-1}(x)$ is **not** $\dfrac{1}{f(x)}$.

$\dfrac{1}{f(x)} = f(x)^{-1}$

E.g. $f(x) = x^2$. domain $= \mathbb{R} =$ codomain.
then f is not surjective ($f(x) \geq 0$ $\forall x$)
not injective.
e.g. $f(1) = f(-1) = 1$

so f^{-1} doesn't exist.

E.g. $f(x) = x^2$. domain $= [0, \infty) =$ codomain.
then f is bijective.
and $f^{-1}(y) = \sqrt{y}$



2. The domains and ranges of f and f^{-1} are interchanged.

3.
$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$
$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or } \cancel{\text{range}} \text{ of } f\text{)}$$
codomain

4. Only a one-to-one function can have an inverse.

Example 5.1.5.

1.
$$\begin{cases} y = e^x, \\ x = \ln y. \end{cases} \qquad x \in \mathbb{R}, y > 0$$
are inverse functions of each other.

a^x, $\forall a > 0$ and $\log_a x$ are inverse functions of each other.

2.
$$\begin{cases} y = x^2, \\ x = \sqrt{y}. \end{cases} \qquad x > 0, y > 0$$
are inverse functions of each other.

3. $y = x^2$, $x \in \mathbb{R}, y \geq 0$ does not have inverse function because it is not one-to-one.

Question: What is the relation between derivatives of inverse functions?

Suppose $y = f(x)$ has an inverse function, then

$$x = f^{-1}(f(x)).$$

$y = f(x)$

$$1 = \frac{dx}{dx} = \frac{df^{-1}}{dy}\frac{dy}{dx}$$

Differentiate both sides with respect to x to get:

$$1 = (f^{-1})'(y) \cdot f'(x)$$

$$\iff \quad \boxed{(f^{-1})'(y) = \frac{1}{f'(x)},}$$

regard x as a function of y.
$= f^{-1}(y)$

or equivalently,

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.}$$

regard $y = f(x)$ as a function of x

Example 5.1.6. Use the identity $\dfrac{d}{dx}e^x = e^x$ to show that

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Solution. Let $y = f(x) = \ln x$. Then its inverse function is $x = e^y$. $\quad \frac{dx}{dy} = e^y$

$$\frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y}.$$

Express the right hand side in terms of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Or, using implicit differentiation: Differentiate the equation $x = e^y$ on both sides with respect to x. We get:

$$1 = \frac{d}{dx}(e^y) = e^y \cdot \frac{dy}{dx} \qquad \text{(the chain rule)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{e^y} = \frac{1}{x}.$$

■

Example 5.1.7. Show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Solution. Let $y = \sqrt{x}$, then $x = y^2$. We have:

$$\frac{d\sqrt{x}}{dx} = \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{2y}.$$

Expressing the right hand side in terms of x, we have

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}.$$

■

Example 5.1.8. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^3 + 4x$.

$f'(x) = 3x^2 + 4$

$f(1) = 5$

1. Find $\dfrac{d}{dx} f^{-1}(x)$ without writing down an explicit formula for $f^{-1}(x)$.
2. Find $\left.\dfrac{d}{dx} f^{-1}(x)\right|_{x=5}$.

Solution.

1. Let $y = f^{-1}(x)$, i.e., $x = f(y)$. Then

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2 + 4}.$$

Alternatively, differentiate both sides of the equation $x = y^3 + 4y$ with respect to x, regarding x now as an implicit function of y. We get:

$f^{-1}(5) = 1$

$$\frac{dx}{dy} = 3y^2 + 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2 + 4}.$$

2. When $x = 5$, $y = f^{-1}(5) = 1$. (Check that $f(1) = 5$!) So,

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=5} = \left.\frac{1}{3y^2 + 4}\right|_{y=1} = \frac{1}{7}.$$

■

5.2 Higher Order Derivatives

Suppose that an object is moving along a coordinate line, and let t denote the time. parametrized by t. Let

$$s = s(t)$$

denote the coordinate of the object at time t. The *velocity* (or "instantaneous velocity") of the object at time t is:

$$v(t) = s'(t).$$

The *acceleration* of the object at time t is:

$$a(t) = v'(t) = s''(t).$$

Notation Let $y = f(x)$.

1st derivative of f: $\quad \dfrac{dy}{dx} = \dfrac{df}{dx} = f'(x)$

2nd derivative of f: $\quad \dfrac{d^2y}{dx^2} = \dfrac{d^2f}{dx^2} = f''(x) = \dfrac{df'(x)}{dx}$

$\vdots \qquad \vdots \qquad \dfrac{d^3y}{dx^3} = \dfrac{d}{dx}f''(x) = f'''(x)$

n-th derivative of f: $\quad \dfrac{d^ny}{dx^n} = \dfrac{d^nf}{dx^n} = f^{(n)}(x)$

Example 5.2.1.

$$\frac{d}{dx}(a^x) = (\ln a)\, a^x$$

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}\left((\ln a)\, a^x\right) = \ln a \frac{d}{dx} a^x = (\ln a)(\ln a) \cdot a^x$$

1.
$$\frac{d^n}{dx^n}(e^x) = e^x, \quad \frac{d^n}{dx^n}(a^x) = a^x \cdot (\ln a)^n.$$

2. $y = x^n, n \in \mathbb{N}$.

$y' = n x^{n-1}$

$y'' = n(n-1)\, x^{n-2}$

$$y^{(m)} = \begin{cases} n(n-1)(n-2)\cdots(n-m+1)x^{n-m}, & \text{if } m < n, \\ n(n-1)(n-2)\cdots 2 \cdot 1 = n!, & \text{if } m = n, \\ 0, & \text{if } m > n. \end{cases}$$

$$\frac{d^{m-n}}{dx^{m-n}}\left(\frac{d^n y}{dx^n}\right) = \frac{d^{m-n}(n!)}{dx^{m-n}}$$

Example 5.2.2. Let y be defined implicitly by the equation $x^2 + y^2 + e^{xy} = 2$. Find y' and y'' at $x = 1$.

$1 + y(1)^2 + e^y = 2$

$y(1) = 0$

Solution. Differentiate both sides of the preceding equation with respect to x to get

$$2x + 2yy' + e^{xy}(y + xy') = 0. \quad ----(1)$$

Then differentiate both sides of the equation with respect to x one more time to get

$$2 + 2y'y' + 2yy'' + e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'') = 0. \quad ----(2)$$

plug in $x = 1, \; y = 0$

$2 + 0 + 1 \cdot (y') = 0$

Inserting $x = 1, y = 0$ into Equations (1), (2), we have:

$$y'|_{x=1} = -2,$$
$$y''|_{x=1} = -10.$$

plug in $x = 1$, $y = 0$, $y' = -2$, in this formula

■

Example 5.2.3. Suppose that $y = e^{\lambda x}$ satisfies $y'' - 2y' - 3y = 0$ (a "differential equation"). Find the constant λ.

Solution. $y = e^{\lambda x}$ implies that $y' = \lambda e^{\lambda x}$, which in turn implies $y'' = \lambda^2 e^{\lambda x}$.

Combining the preceding identities with the equation $y'' - 2y' - 3y = 0$, we have:

$$(\lambda^2 - 2\lambda - 3)e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$ for all x,

$$\lambda^2 - 2\lambda - 3 = 0, \rightarrow \lambda = -1, 3.$$

■

More generally, if $y = e^{\lambda x}$ solves

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

then

$$(a_n \lambda^{(n)} + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0,$$

$\Rightarrow$

$$a_n \lambda^{(n)} + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0 = 0.$$

Exercise 5.2.1. Find λ such that $y = e^{x}$ (λx) satisfies $y''' - 2y'' - 3y' = 0$.
Answer: $\lambda = -1, 0, 3$.

plug in , λ is a constant.

$y = e^{\lambda x}$
$y' = \lambda e^{\lambda x}$
$y'' = \lambda^2 e^{\lambda x}$
$y''' = \lambda^3 e^{\lambda x}$

$\lambda^3 e^{\lambda x} - 2\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} = 0$

$\underbrace{e^{\lambda x}}_{\neq 0}(\lambda^3 - 2\lambda^2 - 3\lambda) = 0 \Rightarrow \lambda^3 - 2\lambda^2 - 3\lambda = 0$

$\lambda(\lambda^2 - 2\lambda - 3)$